

Traveling Faster than the Speed of Light in Non-Commutative Geometry

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Abstract

We study various dynamical aspects of solitons in non-commutative gauge theories and find surprising results. Among them is the observation that the solitons can travel faster than the speed of light for arbitrarily long distances.

1 Introduction

The study of solution to the classical equation of motion of field theories on non-commutative geometries has led to the discovery of new class of soliton solutions [1] which have masses of the order of the inverse of the non-commutativity scale. In the commutative limit, these solitons become infinitely massive and decouple from the theory. In the case of gauge theories, these new soliton solutions are localized magnetic vortices [2, 3, 4, 5, 6]. One remarkable aspect of these vortex solutions is the fact that they can be treated exactly and explicitly by exploiting the algebraic properties of gauge transformations. By now, there are systematic accounts of the solution generating technique and the study of the small fluctuations of these vortices in non-commutative gauge theories in the literature [2, 3, 4, 5, 6].

In this article, we investigate the dynamics of these vortex solutions beyond the small fluctuation approximation. Specifically, we investigate the collective dynamics and the quantum effects of magnetic vortices in 3+1 dimensional NCYM with space-space non-commutativity. To address the nature of the collective dynamics, we construct the generalization of the vortex solutions to solutions with finite velocity along the non-commutative coordinates. The generalization is simple, but should not be dismissed as a trivial boost of the static solution as in the case of solitons in ordinary field theories. In non-commutative theories, boosting along the non-commutative direction is not a symmetry since it acts non-trivially on the non-commutativity parameter $\theta_{\mu\nu}$. Nonetheless, we will find an explicit solution describing a vortex moving at some finite velocity. Moreover, we construct solutions which describe vortices moving with respect to each other. We find that these vortices travel at velocities which can exceed the speed of light.

To study the effect of quantum corrections, we exploit two independent techniques: perturbation theory and the supergravity dual [7, 8]. In formulating the problem of calculating the effect of integrating out some of the degrees of freedom perturbatively at one-loop, we encounter an extremely useful observation: this calculation is equivalent verbatim to some of the calculations done earlier in the context of scattering gravitons and membranes in Matrix theory [9, 10]. By making simple change in the notation, we are able to read off the results of these authors to draw interesting conclusions about the quantum effects on the dynamics of non-commutative vortices. In particular, some of the flat directions in the parameter space of classical non-commutative vortices are lifted by the quantum effects.

The description of non-commutative gauge theories in terms of the supergravity dual can be used to study the dynamics at very large 't Hooft parameter where the quantum effects dominate. The non-commutative vortices correspond to D-string probes in the dual supergravity background. Therefore, the effective dynamics of the non-commutative vortices

is captured by the DBI action. We conclude also using this formalism that these vortices can move faster than the speed of light.

The D-string probe in this supergravity background behave in many ways like the “long string” in AdS_3 [11, 12, 13]. The D-string probe feels a potential which becomes flat near the boundary at $U = \infty$. This potential is consistent with the quantum corrected potential on the moduli space computed perturbatively. The value of the potential at $U = \infty$ is the same as the mass of the classical solution. Just as in the case of the long strings in anti de-Sitter spaces, we conclude that the theory contains a continuum of states above this gap. We also discuss the implications of that conclusion on the different phases of non-commutative gauge theories.

After the completion of this paper, we realized the fact that a non-commutative vortex can travel faster than the speed of light was also noted in [14]. Unlike [14], however, we take the point of view that this phenomenon is real, and provide its explanation in terms of string theory.

2 Non-Commutative Vortex Strings

In this section we describe the construction of magnetic strings in non-commutative gauge theories in four dimensions. To be specific, we will concentrate on $\mathcal{N} = 4$ supersymmetric NCSYM which has six adjoint scalars in addition to the gauge fields. Non-commutative coordinates are taken to be along the (x_2, x_3) -plane. We will begin by reviewing the construction of the static solutions. These solutions can easily be generalized to the constant velocity solutions.

2.1 Static solutions

To describe the non-perturbative solution of non-commutative gauge theories, it is more convenient to work in the operator formalism. Following the notation of [4], let us define the complex coordinates

$$z = \frac{1}{\sqrt{2}}(x^2 + ix^3), \quad \text{such that} \quad [z, \bar{z}] = \theta. \quad (2.1)$$

We will compactify the x_1 direction on a circle. Then, considering configurations which do not depend on x_1 and working in the temporal gauge $A_0 = 0$, the action takes the form

$$S = \frac{2\pi\theta L}{g_{YM}^2} \int dx_0 \text{Tr} \left(-\partial_t \bar{C} \partial_t C + ([C, \bar{C}] + 1/\theta)^2 \right), \quad (2.2)$$

where $C = -A_z + a^\dagger$ and L is the period of compact circle in the x_1 direction.

The equations of motion are

$$\partial_t^2 C = [C, [C, \bar{C}]], \quad (2.3)$$

while gauge fixing constraint yields

$$[C, \partial_t \bar{C}] + [\bar{C}, \partial_t C] = 0. \quad (2.4)$$

Unlike ordinary gauge theories which do not admit static solutions with non zero magnetic fluxes, there are such solutions to eq.(2.3)

$$C = (S^\dagger)^M a^\dagger S^M + \frac{1}{\theta} \sum_{i=0}^{M-1} l^i |i\rangle \langle i|, \quad (2.5)$$

where l^i 's are arbitrary complex numbers which correspond to the position of the monopoles on the (x_2, x_3) -plane and S is the shift operator

$$S = \sum_{i=0}^{\infty} |i+1\rangle \langle i|. \quad (2.6)$$

The field strength in the z plane is

$$\theta F = [C, \bar{C}] + 1 = \sum_{i=0}^{M-1} |i\rangle \langle i|, \quad (2.7)$$

which implies that the total flux is

$$\frac{1}{2\pi} \int dx_2 dx_3 F_{23} = \theta \text{Tr} F = M. \quad (2.8)$$

It is interesting to note that eq.(2.7) does not depend at all on l_i . At first sight this looks like a contradiction with the statement that l_i 's are the locations of the strings in the (x_2, x_3) -plane since the magnetic field clearly should depend on the location of the strings. However, in non-commutative gauge theories, F does not have a gauge invariant meaning and thus cannot be used to specify the location of the strings. In [15] it was shown that the proper gauge invariant generalization of F is F attached to the open Wilson lines of [16]. By probing the solitons (2.5) with these operators, one finds indeed that l_i 's are the locations of the strings [6]. An alternative way to reach the same conclusion is to consider the masses of the small fluctuations [4].

The energy of the solution (2.5) is

$$E = \frac{2\pi\theta}{2g_{YM}^2} \text{Tr} F_0^2 = \frac{\pi LM}{g_{YM}^2 \theta}. \quad (2.9)$$

Notice that the total energy does not depend on l_i . This means that classically there are no static forces between the strings even though they are non-BPS objects.

The $\mathcal{N} = 4$ theory also contains six scalars in the adjoint which enlarge the classical moduli space [4]. Adding

$$\varphi^a = \sum_{i=0}^{M-1} \varphi_i^a |i\rangle \langle i| \quad (2.10)$$

does not affect the classical equation of motion, total flux, or energy. However, it does have one very crucial effect. If the expectation value is large enough, the solution becomes classically stable. To be more precise, in the absence of scalar expectation values, there is a tachyon coming from the “1-3” string sector whose mass is $m^2 = -1/\theta$ [4, 6]. In the presence of scalars, the mass gets shifted [4]

$$m^2 = -\frac{1}{\theta} + \varphi^2. \quad (2.11)$$

Therefore, we see that for sufficiently large expectation values, the solutions are *classically* stable. In fact, in the limit $\varphi \rightarrow \infty$, all of the “1-3” strings become infinitely massive, effectively decoupling the “1-1” sector from the “3-3” sector. In this limit, the effective action of the “1-1” sector becomes simply that of $U(M)$ SYM in 1+1 dimensions.

2.2 Constant velocity solutions

Let us now describe a very simple, but yet interesting, generalization to the static solutions. Consider the same solution (2.5), but instead of setting l_i to be a constant, take

$$l_i = l_i^0 + v_i t. \quad (2.12)$$

One can easily check that both the equation of motion (2.3) and the Gauss law constraint (2.4) are satisfied by this generalization. The total magnetic flux is, of course, intact while the total energy gets a kinetic energy contribution coming from the first term in the action

$$\begin{aligned} E &= \frac{2\pi\theta}{g_{YM}^2} \int dx_0 \int_0^L dx_1 \text{Tr} \left(\partial_t \bar{C} \partial_t C + ([C, \bar{C}] + 1/\theta)^2 \right) = \sum_{i=0}^{M-1} (m + mv_i \bar{v}_i), \\ p &= \frac{2\pi\theta}{g_{YM}^2} \int dx_0 \int_0^L dx_1 \text{Tr} \left(\partial_t C ([C, \bar{C}] + 1/\theta) \right) = \sum_{i=0}^{M-1} mv_i, \\ m &= \frac{\pi L}{g_{YM}^2 \theta}. \end{aligned} \quad (2.13)$$

These equations lead to several remarkable observations about the non-commutative solitons. First, we see that not only are there no static forces between the strings, there are no forces which depend on the velocities of the strings either. Therefore, at least classically, the

vortices propagate like free particles. Second, and more importantly, the kinematics of the vortices are *non-relativistic*.¹ We will discuss this point further in the concluding section.

3 Quantum Effective Dynamics

In the previous section, we saw that the non-commutative magnetic solitons admit different surprising properties. The aim of this section is to see how much of this survives the quantum corrections. The two methods at our disposal are perturbation theory which is valid at small coupling and the supergravity description which is valid at large coupling.

3.1 Perturbation theory

For simplicity, we consider only one unit of magnetic flux. In that case, the interesting part of the moduli space is parameterized by φ^a . As was briefly reviewed in the previous section, the form of the potential of small fluctuations about the string solution is

$$V = T^2 \left(-\frac{1}{\theta} + \varphi^2 \right), \quad (3.1)$$

where T represents fluctuations coming from the “1-3” strings. What we would like to do is to integrate out T to induce an effective potential for φ . This cannot be done for small φ due to the presence of the tachyon. Therefore, we will focus our attention on finding the effective potential for $\varphi^2 > 1/\theta$.

Fortunately, this somewhat tedious calculation was done earlier in the context of scattering gravitons off membranes in Matrix theory [9, 10]. Modulo the trivial modification between the 0-brane soliton in 2+1 dimensions and the 1-brane soliton in 3+1 dimensions, the calculation [9, 10] is verbatim the calculation we wish to do here. All that we need to do is to appropriately smear equation (66) of [10]

$$V = \frac{3N}{16} \frac{c^3}{b^5} \implies V = \frac{LN}{8\pi} \frac{c^3}{b^4} \quad (3.2)$$

(N is the rank of the gauge group) and to appropriately relabel the variables: $c = 1/\theta$, $b = \varphi_0$, to find that the effective potential as a function of φ_0 is

$$V_{eff} = -\frac{(2\pi)^3 LN}{4(2\pi\varphi_0)^4 \theta^3}. \quad (3.3)$$

We see, therefore, that the classical moduli space is lifted by quantum corrections which tend to lower the expectation value of the scalars. If one starts with the vortex at some large but

¹This statement is closely related to the observations regarding the non-relativistic dispersion relations of closed strings in NCOS [17]. See also [18, 19].

finite value of $|\varphi_0|$, the quantum correction will lift the flat direction and the expectation value of $|\varphi_0|$ will start rolling down. Eventually, $|\varphi_0|$ becomes small enough and some of the “1-3” strings become tachyonic, where we must abandon the perturbative calculation all together.

From eq.(3.3) one finds that the lifetime of the soliton is roughly $\varphi_0^3 \theta^2 / \sqrt{\lambda}$, which means that even quantum mechanically, the solitons can live for an arbitrarily long time (by taking φ_0 to be large).

3.2 Supergravity description

The relevant parts of the supergravity background are given by [7, 8]

$$\begin{aligned} \frac{ds^2}{\alpha'} &= \frac{U^2}{\sqrt{\lambda}}(-dx_0^2 + dx_1^2) + \frac{\sqrt{\lambda}U^2}{\lambda + \Delta^4 U^4}(dx_2^2 + dx_3^2) + \frac{\sqrt{\lambda}}{U^2}dU^2 + \sqrt{\lambda}d\Omega_5^2 \\ e^\phi &= \frac{g_{YM}^2}{2\pi} \sqrt{\frac{\lambda}{\lambda + \Delta^4 U^4}}, \quad B_{23} = -\frac{\alpha' \Delta^2 U^4}{\lambda + \Delta^4 U^4}, \quad A_{01} = \frac{2\pi}{g_{YM}^2} \frac{\alpha' \Delta^2 U^4}{\lambda} \end{aligned} \quad (3.4)$$

where $\lambda = 2g_{YM}^2 N$ and $\theta_{23} = \theta = 2\pi\Delta^2$.

The dual of the magnetic string considered in the previous sections is a D1-brane oriented along x_0 and x_1 . The action of such a D-string in the background (3.4) is

$$V(U) = \frac{L}{g_{YM}^2} \left(\frac{U^2 \sqrt{\lambda + \Delta^4 U^4}}{\lambda} - \frac{\Delta^2 U^4}{\lambda} \right). \quad (3.5)$$

Simple inspection near $U = 0$ gives

$$V(U) = \frac{LU^2}{g_{YM}^2 \sqrt{\lambda}} + \mathcal{O}(U^6), \quad (3.6)$$

whereas at infinity

$$V(\infty) = \frac{L}{2g_{YM}^2 \Delta^2}. \quad (3.7)$$

The important point is the fact that this potential is finite. The potential which starts out growing quadratically in U becomes flat at large U and converge to (3.7). See figure 1 for an illustration. Because the potential becomes flat in the $U \rightarrow \infty$ limit, the brane sitting at $U = \infty$ is a solution to the equations of motion. Such a configuration of strings winding along the boundary of space-time at finite cost in energy is known as the “long string” and has appeared in several contexts of AdS/CFT correspondence [11, 12, 13]. Note that in the commutative limit, the quadratic dependence (3.6) persists for all values of U , and the mass of the would-be long D-string becomes infinite.

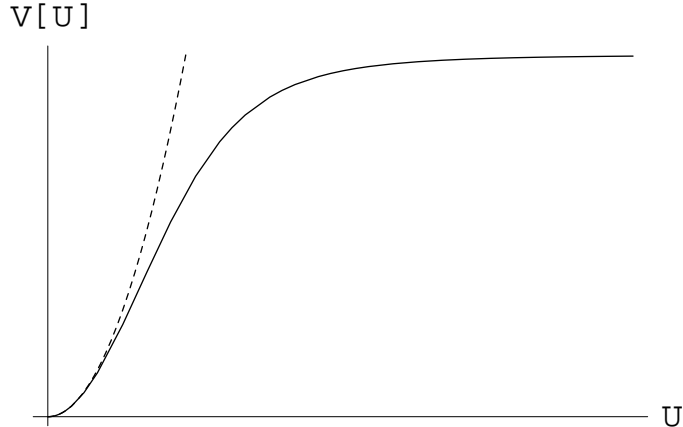


Figure 1: The D1-branes potential as a function of the radial coordinate. The solid line is the potential in the non-commutative case in which with a finite amount of energy the D1-brane can reach the boundary. The dotted line is the potential in the commutative case.

The form of the potential illustrated in figure 1 suggests that a D-string which starts in the neighborhood of $U = 0$ can “escape to infinity” if it carries enough kinetic energy. The minimum energy required to escape to infinity is the gap (3.7). At energies below this gap, the D-string will be bounced back to the near horizon region, whereas at energies above this gap, the D-string will escape to infinity. We emphasize, however, that it will take infinite time for the brane to reach the boundary. This statement is the same (up to time reversal) as the statement of the previous section that the lifetime of the soliton can be made arbitrarily long. The kinetic energy of the string escaping to infinity is therefore not quantized, giving rise to a continuum of states analogous to what was seen in the case of AdS_3 .

Note that the mass of the long string (3.7) is *exactly* the same as the mass found in the free theory limit (2.9). This is surprising since these solutions are non-BPS, and in general there is no reason for their masses to be protected.² The reason why they are is the fact that at large U (compared to $1/\Delta$) the supergravity solution looks like the near horizon region of D1-branes oriented along (x_0, x_1) and smeared along the (x_2, x_3) plane. A D1 probe with the same orientation is of course BPS and thus the potential at infinity is protected. We will find more evidence for this claim in the discussion below.

In the previous section, we saw that the quantum corrections lifted the potential by introducing the term (3.3). To see this term in supergravity, we expand the exact supergravity

²For example, in [20] a holographic description of some non-BPS branes was found to receive correction to their masses due to strong coupling.

potential for the D1-brane around $U = \infty$ and find

$$V(U) = \frac{L}{2\Delta^2 g_{YM}^2} - \frac{NL}{4\Delta^6 U^4} + \mathcal{O}(U^{-8}) \quad (3.8)$$

in exact agreement with the field theory result (3.3) with the usual identification [21] $U = |2\pi\varphi_0|$.

While such a nice agreement with the weakly coupled field theory results is found at large U , at small U the strongly coupled description implies a completely different physics. Recall that in perturbation theory, we encountered a tachyon at small φ (or small U) coming from the “1-3” strings. For large ’t Hooft coupling, we do not encounter such a behavior at small values of U . The background geometry simply becomes that of $AdS_5 \times S_5$, and the D-string will simply continue to fall toward the horizon in this background. In this region, the physics of the falling D-string is identical to the one described in [22, 23]. From any finite U , it takes infinite time for the D-string to reach the horizon. The falling of the brane is to be interpreted as the spreading of the flux at approximately the speed of light.

In AdS_3 the effective description of the long string is the Liouville theory [12] which plays an important role in the duality. To find the effective description of the long string in our case we consider fluctuations of the D1-brane:

$$\begin{aligned} F_{01} &= \partial_0 A_1(t, x_1) - \partial_1 A_0(t, x_1), \\ x_2 &= 2\pi\Delta^2 \varphi_2(t, x_1), \\ x_3 &= 2\pi\Delta^2 \varphi_3(t, x_1), \\ U &= U_0 + 2\pi\varphi_4(t, x_1), \\ \vec{\Omega}_5 &= \frac{2\pi}{U_0} \varphi_i(t, x_1), \quad i = 5 \dots 10. \end{aligned} \quad (3.9)$$

Substituting this into the DBI action and taking the $U_0 \rightarrow \infty$ limit leads to an effective action for the fluctuations given by

$$S = \frac{1}{2\Delta^2 g_{YM}^2} - \frac{(2\pi\Delta)^2}{g_{YM}^2} \left(\frac{1}{4} F^2 + \sum_{i=2\dots 10} \frac{1}{2} (\partial\varphi_i)^2 \right), \quad (3.10)$$

which is precisely the bosonic part of the action of ordinary supersymmetric Yang-Mills theory in 1+1 dimensions with 16 supercharges and the two dimensional coupling constant $g_{YM}^2/2\pi\theta$. It is worthwhile to emphasize that the quadratic form of the action is not a result of taking the field strength to be small. On the contrary, the DBI correction to the action is controlled by F^2/U_0^4 , and in the limit $U_0 \rightarrow \infty$ keeping F fixed, the action becomes exactly SYM. In other words, $1/U_0^2 \rightarrow 0$ replaces the role of $\alpha' \rightarrow 0$ of the usual decoupling limit. This is the same as the $\varphi_0 \rightarrow \infty$ limit which decouples the “1-1” sector from the “3-3” sector described in the previous subsection.

It is also very easy to see in the supergravity dual that the D1-brane can move faster than the speed of light. Consider the dependence of the D1-brane action on the velocity $v^2 = (\partial_0 x_2)^2 + (\partial_0 x_3)^2$

$$S = \frac{\Delta^2 U_0^4}{\lambda g_{YM}^2} \left(\sqrt{1 + \frac{\lambda}{\Delta^4 U_0^4}} \sqrt{1 - \frac{v^2}{1 + \Delta^4 U_0^4 / \lambda}} - 1 \right). \quad (3.11)$$

We see that at large U_0 the bound on the velocity of the string, as measured in the field theory units, is not 1 (the speed of light) but rather $U_0^2 \theta$ which goes to infinity. Moreover, for $U_0 \rightarrow \infty$, eq.(3.11) yields the non-relativistic dispersion relation of (2.13). It should be noted that the velocity of the string as measured by a local observer in the supergravity background is, of course, smaller than the speed of light.

4 Phases of NCSYM

In the previous section, we saw that there are 1+1-dimensional ordinary $U(M)$ SYM theories within 3+1 dimensional NCSYM above a mass gap which is proportional to M . This statement has important implications on the phases³ of NCSYM. In this section we explore this notion and make contact with [25].

First let us recall the different phases of the theory at low energies which we parameterize using the 't Hooft coupling and the S-dual 't Hooft coupling, $\tilde{\lambda} = N^2/\lambda$.

- I** $\lambda \ll 1$ is the weakly coupled NCSYM phase.
- II** $1 \ll \lambda, 1 \ll \tilde{\lambda}$ is the dual supergravity phase.
- III** $\tilde{\lambda} \ll 1$ is the weakly coupled phase of the S-dual theory.

These regions are non-overlapping. The intermediate region **II** exists only for large N . The S-dual theory in the region **III** is the NCOS theory [26, 27, 28]. Although NCOS is a valid description at low energies, one finds a richer phase structure in the theory at higher energies. The basic idea is similar to what was discussed in [24], and was partially discussed already in [29].

One way to see that the theory goes into a new phase at higher energies is to note that the dual supergravity description remains valid even in region **III** for sufficiently high temperatures. The reason is that in this supergravity background, the dilaton is U dependent and goes to zero for large U . Note that this is very different from the behavior of the commutative theory. Among other things, this means that the supergravity description is

³We use the word “phase” here in the sense of [24]. The phase boundaries in this context may or may not be a phase transition in the sense of non-analyticities in the thermodynamic observables.

applicable in the UV even for small values of N , *even for* $U(1)$, as long as we take the coupling λ to be large enough! It is easy to verify that the range of validity⁴ of the supergravity description is $U \gg \sqrt{\lambda/N\theta}$, or equivalently

$$T \approx \frac{U}{\sqrt{\lambda}} \gg \frac{1}{\sqrt{N}\Delta}. \quad (4.1)$$

Note that since a theory is defined at the fundamental level in the UV (where the number of degrees of freedom is maximal), what we are saying is that at the fundamental level the definition of the theory is in terms of the supergravity dual. The NCOS phase is an effective description valid *only* in the deep IR.

The NCOS theory in region **III** undergoes a Hagedorn phase transition [25] at temperature

$$T_H = \frac{1}{2\pi\alpha_{eff}} = \frac{1}{g_{YM}\sqrt{\theta}}. \quad (4.2)$$

The perturbative NCOS description is valid below this temperature, but above T_H , the system is better described as a thermal ensemble of free strings. The physics of these strings resembles that of the DVV strings [30]. The complete picture of the phase structure can be summarized in a phase diagram. See figure 2.

The Hagedorn transition is a phenomenon of weakly coupled NCOS theory in region **III**. When the NCOS coupling constant $\tilde{\lambda}$ is taken to be sufficiently large, one finds oneself in region **II**, where there is no Hagedorn transition. Therefore, supergravity is the proper description of the theory in region **II** for all energy scales. We also expect to see that the liberated strings re-condense at temperatures above the DVV/SUGRA phase boundary (4.1) in region **III**. In other words, there should be no overlap between the DVV and the SUGRA phases. Let us examine this assertion more closely.

First, let us recall the basic mechanism of [25] which drives the Hagedorn phase transition in region **III**. NCOS is a certain decoupling limit of D3-F1 bound state. There is an energy gap to liberate the F1 from the D3. However, at temperatures above T_H , the entropic contribution will compensate the free energy, making the configuration of liberated F1 strings the preferred state of the system.

⁴A supergravity description is valid when both the dilaton and the curvature in string units are small. Just like in the case of $AdS_5 \times S^5$, the curvature of the background (3.4) is of the order of $1/\sqrt{\lambda}$ and is therefore small in region **III**. One important distinction of the non-commutative case is the fact that the dilaton is not constant but rather goes to zero at infinity. Hence, the supergravity description of NCSYM is valid even in region **III** for sufficiently large U . For smaller values of U , the dilaton starts growing and eventually we have to apply S-duality. In the S-dual background, the dilaton is getting smaller as we go further in the IR, but the curvature starts growing and becomes of order one at the boundary (4.1). This is similar to the phase diagram of D5-branes [24].

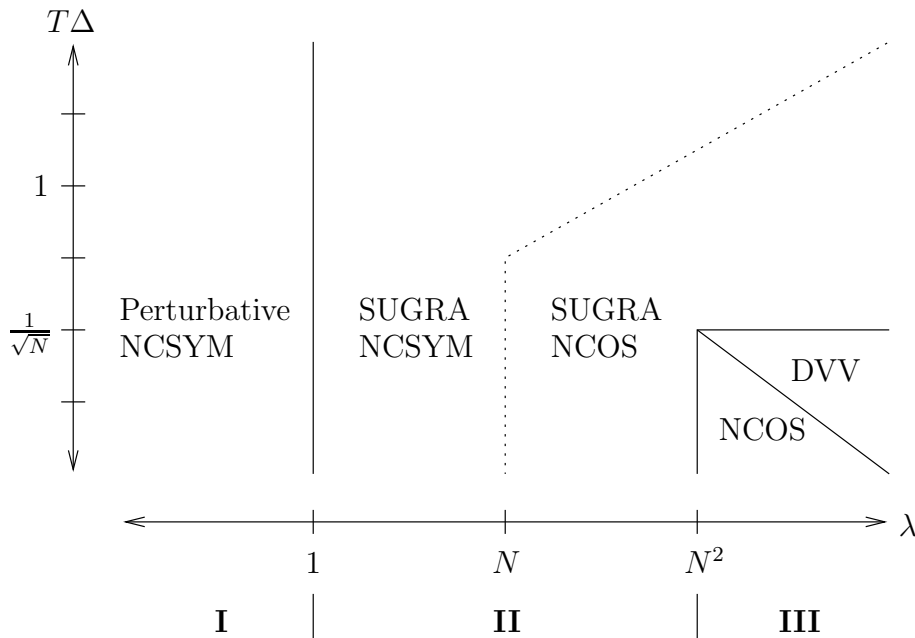


Figure 2: Phase diagram of NCSYM. The horizontal axis parameterizes the 't Hooft coupling and the vertical axis parameterizes the temperature. Both axes are logarithmic.

In order to argue that the Hagedorn transition ceases to occur in region **II**, it suffices to show that one does not gain in free energy by liberating the F1. Note that the F1 in NCOS is dual to the D1 probe in the supergravity description of NCSYM which we considered in the previous section. Therefore, what we have to do is to compute the effect of the finite temperature on the mass gap (3.7). This can be inferred from the action of the string probes in the finite temperature generalization of the supergravity background. The only relevant effect of the temperature⁵ is to modify the supergravity background by multiplying g_{00} by a factor $\left(1 - \frac{U_0^4}{U^4}\right)$ where $U_0 = T\sqrt{\lambda}$. This will give rise to a new expression for the energy gap

$$E_{gap} = \frac{L}{g_{YM}^2 \Delta^2} + LT^4 \theta N, \quad (4.3)$$

which states that the energy gap increases as we increase the temperature. The negative contribution from the entropy to the free energy remains the same as in [25]. Therefore the free energy per unit length of the string probe is

$$f = \frac{F}{L} = \frac{E - ST}{L} = \frac{1}{g_{YM}^2 \Delta^2} + T^4 \theta N - T^2. \quad (4.4)$$

The strings are liberated when $f \leq 0$. One can easily see that this is possible only in region **III** and hence the DVV phase is not realized in region **II**. Moreover, even in region **III** the

⁵We are not careful here with numerical factors of order 1.

DVV description is realized only for

$$\frac{1}{\sqrt{\theta N}} > T > \frac{1}{g_{YM}\sqrt{\theta}}, \quad (4.5)$$

which is in exact agreement with eqs.(4.1) and (4.2).⁶

5 Conclusion

In this article, we considered various dynamical aspects of solitons in non-commutative gauge theories. The most dramatic and surprising observation of this study is the fact that these solitons can travel faster than the speed of light. We will comment on this remarkable result in the remainder of this paper.

Traveling faster than the speed of light is, of course, not possible in a Lorentz invariant theory. If Lorentz invariance is broken, one might naively expect to be able to travel faster than the speed of light for at most the distances of the order of the scale of broken Lorentz symmetry. In nature, this distance is very small since Lorentz symmetry is tested to hold to very small length scales. The result we obtain in this paper illustrates that this picture is too naive. Non-commutative gauge theories with $\mathcal{N} = 4$ supersymmetry breaks Lorentz invariance only for length scales smaller than the non-commutativity scale. Yet, there is no bound on the distances the non-commutative solitons can travel at speeds faster than the speed of light.⁷ This is remarkable especially in light of the fact that we have not made the time coordinate non-commutative.

One might worry that being able to travel faster than the speed of light will cause causality to break down. Indeed, a signal traveling faster than the speed of light looks to a moving observer like a signal traveling backwards in time. However, problems with causality arise only if an observer can send a signal from the future to itself in the past. This turns out not to be possible, essentially because there is a preferred frame in which the information can travel arbitrarily faster than the speed of light, but not backwards in time. This is the frame of the stationary observer. Any signal transmitted by an observer can only be received by the same observer in the future. See figure 3 for more details.

Along similar lines, one can see that problems with unitarity do not arise. Non-commutative geometry as seen by a moving observer is certainly a strange world. The action has infinitely many time derivatives, and a Hamiltonian cannot be defined in general. However, precisely

⁶Similar conclusion, using a different approach, was reached by J. Barbon and E. Rabinovici (to appear).

⁷A different mechanism for transmitting signals faster than the speed of light, which relies on broken Lorentz invariance at large distances as a result of UV/IR mixing [31] in a theory with less supersymmetry, was discussed in [32].

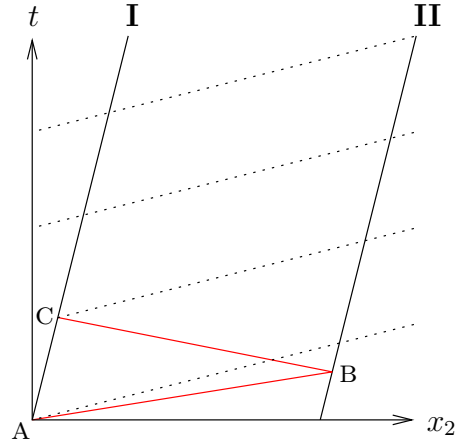


Figure 3: “Back to the Future.” Two observers, **I** and **II**, are in motion relative to the preferred frame (t, x_2) . The dashed lines represent the constant time slices in the frame of **I** and **II**. **I** can send a signal from A which **II** receives at a point B in the past. However, when **II** sends the information back to **I** it is at the point C which is always in the future of A as seen by **I**.

because there is a preferred frame in which non-commutativity is strictly space-like, a Hamiltonian *can* be defined in that frame. In other words, the criterion of [33] is not violated by theories with only space-like non-commutativity.

While propagation at speeds faster than the speed of light may seem strange from the conventional field theory points of view, its origin can be understood in very simple terms in the language of the underlying string theory. The “speed of light” is defined as the speed at which the “3-3” strings propagate in the *open string* metric since the large B -field along the D3-brane affects their dynamics. The non-commutative solitons, on the other hand, are D1-branes. The large B -field is transverse to the D1-brane, and consequently does not affect the dynamics of the “1-1” strings. Therefore, the D1-branes are living in the *closed string* metric. As was explained in [34, 35], the ratio between the closed string metric and the open string metric goes to zero in the field theory limit. Therefore, the relativistic dispersion relation in the closed string metric becomes the non-relativistic dispersion relation in the open string coordinates in this limit. This should also have important implications for coupling non-commutative field theories with gravity.

Conventional ideas for traveling faster than the speed of light involve “short-cuts” in space-time such as the wormholes and the warps. The result presented in this paper offers a new alternative: non-commutatizing the universe. Or better yet, nature may already be non-commutative at small length scales. One can imagine a scenario where one is living on a brane with a large background B -field. One can then transmit signals at speeds faster than

the speed of light by sending it in the bulk (closed string) metric. It should be amusing to study this further.

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